5.2

13) Call A the matrix.

$$\det(A-\lambda I) = \begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 9-\lambda & 0 \\ 5 & 8 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\cdot\begin{vmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{vmatrix} = (3-\lambda)\big((6-\lambda)(9-\lambda)-4\big)$$

$$= (3-\lambda)(\lambda^2 - 15\lambda + 50)$$ is the characteristic polynomial.

5.3

11) Call E_λ eigenspace of e. value λ, A the matrix.

$\lambda = 1, 2, 3$

$$A - 1I = \begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -5 & 5 & 0 \end{bmatrix} \sim$$

1.

$$\sim \begin{bmatrix} -1 & 2 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Get $x_1 = x_2$

$\therefore \quad x_3 = 2x_2 - x_1 = x_1$

$\therefore \quad E_1 = \left\{ c \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ; c \in \mathbb{R} \right\}$

$$A - 2I = \begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -3 & 2 & 0 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} -3 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \therefore$$

Get $x_3 = x_2 \;, \; x_1 = \frac{2}{3} x_2$

$E_2 = \left\{ c \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} ; c \in \mathbb{R} \right\}$

$$A - 3I = \begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 2 & -1 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad x_2 = 3x_1 \; ; \; x_3 = 2x_2 - 2x_1 = 4x_1 \; ; \; \therefore E_3 = \left\{ c \cdot \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} ; c \in \mathbb{R} \right\}$

Diagonalization: $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$; $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$A = PDP^{-1}$

5.4

9) a) $T(5+3t) = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

b) Take P_1, P_2 in $\mathbb{P}_2$ an c in $\mathbb{R}$

$$T(P_1+P_2) = \begin{bmatrix} (P_1+P_2)(-1) \\ (P_1+P_2)(0) \\ (P_1+P_2)(1) \end{bmatrix} = \begin{bmatrix} P_1(-1) + P_2(-1) \\ P_1(0) + P_2(0) \\ P_1(1) + P_2(1) \end{bmatrix} =$$

$$= \begin{bmatrix} P_1(-1) \\ P_1(0) \\ P_1(1) \end{bmatrix} + \begin{bmatrix} P_2(-1) \\ P_2(0) \\ P_2(1) \end{bmatrix} = T(P_1) + T(P_2).$$

$$T(cP_1) = \begin{bmatrix} cP_1(-1) \\ cP_1(0) \\ cP_1(1) \end{bmatrix} = c\begin{bmatrix} P_1(-1) \\ P_1(0) \\ P_1(1) \end{bmatrix} = c \cdot T(P_1)$$

c) $T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $T(t) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $T(t^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$M_T = \left[T(1) \; ; T(t) \; ; T(t^2) \right] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$